Unbiased test and unbiased critical region

Let us consider the testing of $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$.

A test based on critical region W is said to be <u>unbiased</u> if the power of the test $\geq$ size of the critical region.

$$\text{power} = 1-\beta \geq \alpha$$
$$\Rightarrow P_{\theta_1}(W) \geq P_{\theta_0}(W)$$

For $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$
unbiasedness gives
$$P_{\theta}(W) \geq P_{\theta_0}(W) \quad \forall\ \theta (\neq \theta_0) \in \Theta$$

<u>Result</u>- Every most powerful or uniformly most powerful (UMP) critical region is necessarily unbiased.

<u>Proof</u>. Suppose W is MP critical region of size α for testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$, by N-P lemma.

critical region $\rightarrow$ $W = \{ \underline{x} : L(\underline{x}, \theta_1) \geq K L(\underline{x}, \theta_0) \}$

acc. region $\rightarrow$ $A = \{ \underline{x} : L(\underline{x}, \theta_1) < K L(\underline{x}; \theta_0) \}$

where K is determined from the size condition α.

$$\alpha = \int_W L_0\, dx = P(\underline{x} \in W / H_0)$$

$$\boxed{\text{To show } P_{\theta_1}(W) \geq \alpha}$$

$$P_{\theta_1}(W) = \int_W L_1\, d\underline{x} \geq K \int_W L_0\, d\underline{x} = K\alpha$$
[From N-P lemma.]

$$\therefore P_{\theta_1}(W) \geq K\alpha \quad \forall\ K > 0. \qquad \text{—— (i)}$$

Also,
$$1 - P_{\theta_1}(W) = 1 - P(\underline{x} \in W / H_1) = P(\underline{x} \in A / H_1)$$
$$= \int_A L_1\, dx$$
$$< K \int_A L_0\, dx \qquad \text{[From N-P lemma]}$$
$$= K[1 - P(\underline{x} : \underline{x} \in W / H_0)]$$
$$= K(1-\alpha)$$
$$\therefore 1 - P_{\theta_1}(W) < K(1-\alpha) \qquad \text{—— (ii)}$$

Case I   We know K positive.

$K > 1$

Then ~~P_{θ_1}~~ from (i) $P_{\theta_1}(W) \geq K\alpha > \alpha$

$\therefore$ W is unbiased.

Case II $0 < K < 1$, from (ii)

$1 - P_{\theta_1}(W) < K(1-\alpha) < 1-\alpha$

$\Rightarrow$ $P_{\theta_1}(W) > \alpha$

$\therefore$ W is unbiased.

A similar logic leads to prove W is unbiased if ~~g~~ it is uniformly most powerful.

Result. sufficient statistic and W.

Let T be suff. stat. for θ. Then

$$L(\underline{x};\theta) = \prod_{i=1}^{n} f(x_i,\theta)$$
$$= g_\theta(t(\underline{x})).h(\underline{x})$$

By N-P lemma,

$$W = \{\underline{x} : L(\underline{x};\theta_1) \geq K\,L(\underline{x};\theta_0)\} \quad \forall\, K>0$$
$$= \{\underline{x} : g_{\theta_1}(t(\underline{x})).h(\underline{x}) \geq K.\,g_{\theta_0}(t(\underline{x})).h(\underline{x})\} \quad \forall\, K>0$$
$$= \{x : g_{\theta_1}(t(\underline{x})) \geq K\,g_{\theta_0}(t(\underline{x}))\}$$

So a MP test based on $T(x)$ is same as MP test based on the joint distribution. $L(\cdot)$.

Example $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1 > \theta_0$. $X_1, X_2, \cdots, X_n$ be i.i.d. r.v

$$\frac{L_1}{L_0} = \frac{e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\theta_1)^2}}{e^{-1/2\sigma^2\Sigma(x_i-\theta_0)^2}} \geq K.$$

$$\Rightarrow e^{-1/2\sigma^2[2\Sigma x_i(\theta_1-\theta_0)+n(\theta_1^2-\theta_0^2)]} \geq K.$$

$$\Rightarrow 2\Sigma x_i(\theta_1-\theta_0)+n(\theta_1^2-\theta_0^2) \geq \log K.$$

$\Rightarrow$ $\bar{x} > K'$ [As $\bar{x}$ is suff. stat for θ].

Or you can simplify it as well.

For right tail test CR will be $W = \{\underline{x} : \bar{x} > K'\}$

For left tail test CR will be $W = \{\underline{x} : \bar{x} < K''\}$

K' and K'' are constants.